OPERATOR CONNECTIONS AND BOREL MEASURES ON THE UNIT INTERVAL

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Abstract. A connection is a binary operation for positive operators satisfying the monotonicity, the transformer inequality and the joint-continuity from above. A mean is a normalized connection. In this paper, we show that there is a one-to-one correspondence between connections and finite Borel measures on the unit interval via a suitable integral representation. Every mean can be regarded as an average of weighted harmonic means. Moreover, we investigate decompositions of connections, means, symmetric connections and symmetric means.

Key words. operator connection, operator mean, operator monotone function, Borel measure

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1. Introduction.

A general theory of connections and means for positive operators was given by Kubo and Ando [9]. Let $B(H)$ be the von Neumann algebra of bounded linear operators on a Hilbert space H. The set of positive operators on H is denoted by $B(H)^+$. For Hermitian operators $A, B \in B(H)$, the partial order $A \leq B$ means that $B - A \in B(H)^+$. A *connection* is a binary operation σ on $B(H)^+$ such that for all positive operators A, B, C, D:

- (M1) *monotonicity*: $A \leq C, B \leq D$ implies $A \,\sigma\, B \leq C \,\sigma\, D$
- (M2) *transformer inequality*: $C(A \,\sigma\, B)C \leq (CAC) \,\sigma\, (CBC)$
- (M3) *continuity from above*: for $A_n, B_n \in B(H)^+$, if $A_n \downarrow A$ and $B_n \downarrow B$, then $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$. Here, $A_n \downarrow A$ indicates that A_n is a decreasing sequence and A_n converges strongly to A.

Two trivial examples are the left-trivial mean $(A, B) \mapsto A$ and the right-trivial mean $(A, B) \mapsto B$. Typical examples of a connection are the sum $(A, B) \mapsto A + B$ and the parallel sum

$$A : B = (A^{-1} + B^{-1})^{-1}, \quad A, B > 0,$$

the latter being introduced by Anderson and Duffin [1]. A *mean* is a connection σ with normalized condition $I \,\sigma\, I = I$ or, equivalently, fixed-point property $A \,\sigma\, A = A$

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for all $A \geq 0$. The class of Kubo-Ando means cover many well-known operator means in practice, e.g.

- t-weighted arithmetic means: $A \nabla_t B = (1-t)A + tB$
- t-weighted geometric means: $A \#_t B = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}$
- t-weighted harmonic means: $A \,!_t\, B = [(1-t)A^{-1} + tB^{-1}]^{-1}$
- logarithmic mean: $(A,B) \mapsto A^{1/2} f(A^{-1/2}BA^{-1/2})A^{1/2}$ where $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = (x-1)/\log x$.

This axiomatic approach has many applications in operator inequalities (e.g. [3], [6], [12]), operator equations (e.g. [2], [10]) and operator entropy ([5]).

A fundamental tool of Kubo-Ando theory of connections and means is the theory of operator monotone functions. This concept is introduced in [11]; see also [4], [7], [8]. A continuous real-valued function f on an interval I is called an *operator monotone function* if, for all Hermitian operators $A, B \in B(H)$ whose spectrums are contained in I and for all Hilbert spaces H, we have

$$A \leq B \Longrightarrow f(A) \leq f(B).$$

A major core of Kubo-Ando theory is the interplay between connections, operator monotone functions and Borel measures. Note first that if σ and η are connections and $k_1, k_2 \in \mathbb{R}^+ = [0, \infty)$, the binary operation

$$k_1\sigma + k_2\eta : (A,B) \mapsto k_1(A \,\sigma\, B) + k_2(A \,\eta\, B)$$

is also a connection. This shows that the set of connections on $B(H)^+$ forms a cone. Introduce a partial order $\leq$ on this cone by $\sigma \leq \eta$ if and only if $A \,\sigma\, B \leq A \,\eta\, B$ for all $A, B \geq 0$. Equip the cone of operator monotone functions from $\mathbb{R}^+$ to $\mathbb{R}^+$ with the pointwise order, i.e. $f \leq g$ means that $f(x) \leq g(x)$ for all $x \in \mathbb{R}^+$. The cone of finite Borel measures on a topological space is also equipped with the usual partial order. In [9], a connection σ on $B(H)^+$ can be characterized via operator monotone functions as follows:

Theorem 1.1 ([9]). *Given a connection σ, there is a unique operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying*

$$f(x)I = I \,\sigma\, (xI), \quad x \in \mathbb{R}^+.$$

Moreover, the map $\sigma \mapsto f$ is an affine order-isomorphism.

We call f the *representing function* of σ. A connection also has a canonical representation with respect to a Borel measure as follows.

THEOREM 1.2 ([9]). *Given a connection σ, there is a unique finite Borel measure μ on $[0,\infty]$ such that*

$$A\,\sigma\,B = \alpha A + \beta B + \int_{(0,\infty)} \frac{\lambda+1}{\lambda}\{(\lambda A) : B\}\,d\mu(\lambda), \quad A,B \geq 0 \tag{1.1}$$

where the integral is taken in the sense of Bochner, $\alpha = \mu(\{0\})$ and $\beta = \mu(\{\infty\})$. Moreover, the map $\sigma \mapsto \mu$ is an affine isomorphism.

We call μ the *representing measure* of σ. In particular, every connection arises in form (1.1). A connection is a mean if and only if its representing function is normalized (i.e. $f(1) = 1$) or, equivalently, its representing measure is a probability measure.

In this paper, we characterize connections, means, symmetric connections and symmetric means in terms of Borel measures on the unit interval. The main result of this paper is the following:

THEOREM 1.3. *Given a finite Borel measure μ on $[0,1]$, the binary operation*

$$A\,\sigma\,B = \int_{[0,1]} A\,!_t\,B\,d\mu(t), \quad A,B \geq 0 \tag{1.2}$$

is a connection on $B(H)^+$. Moreover, the map $\mu \mapsto \sigma$ is bijective, affine and order-preserving, in which case the representing function of σ is given by

$$f(x) = \int_{[0,1]} 1\,!_t\,x\,d\mu(t), \quad x \geq 0. \tag{1.3}$$

This theorem states that there is an affine isomorphism between connections and finite Borel measures on $[0,1]$ via a suitable form of integral representation. Moreover, a connection σ is a mean if and only if μ is a probability measure. Hence every mean can be regarded as an average of weighted harmonic means. The weighted harmonic means form a building block for general connections. A surprising example is that the dual of the logarithmic mean can be expressed as the integral of weighted harmonic means with respect to Lebesgue measure on $[0,1]$. Recall that a symmetric connection is a connection σ such that $A\sigma B = B\sigma A$ for all $A,B \geq 0$. It follows that every symmetric connection admits an integral representation

$$A\,\sigma\,B = \frac{1}{2}\int_{[0,1]} (A\,!_t\,B) + (B\,!_t\,A)\,d\mu(t), \quad A,B \geq 0$$

for some finite Borel measure μ invariant under $\Theta : [0,1] \to [0,1]$, $t \mapsto 1-t$. The integral representation (1.2) also has advantages in treating decompositions of connections. It is shown that a connection σ can be written as

$$\sigma = \sigma_{ac} + \sigma_{sd} + \sigma_{sc}$$

where σ_{ac}, σ_{sd} and σ_{sc} are connections. The "singular discrete part" σ_{sd} is a countable sum of weighted harmonic means with nonnegative coefficients. The "absolutely continuous part" σ_{ac} has an integral representation with respect to Lebesgue measure m on $[0,1]$. The "singular continuous part" σ_{sc} has an integral representation with respect to a continuous measure mutually singular to m.

Here is the outline of the paper. In Section 2, we prove Theorem 1.3 and its consequences. The integral representations of well-known operator connections with respect to finite Borel measures on $[0,1]$ are also given. Decompositions of connections, means, symmetric connections and symmetric means are discussed in Section 3.

2. Connections and Borel measures on the unit interval. In this section, we investigate the relationship between connections and Borel measures on $[0,1]$. Our main result states that there is an affine correspondence between connections and finite Borel measures on $[0,1]$ via a suitable integral representation. To prove this fact, recall the following:

LEMMA 2.1 ([11]). *A continuous function $f : \mathbb{R}^+ \to \mathbb{R}^+$ is operator monotone if and only if there is a unique finite Borel measure ν on $[0,\infty]$ such that*

$$f(x) = \int_{[0,\infty]} \frac{x(\lambda+1)}{x+\lambda}\, d\nu(\lambda), \quad x \geq 0. \tag{2.1}$$

Proof of Theorem 1.3. Since the family $\{A \,!_t\, B\}_{t\in[0,1]}$ is uniformly bounded by $\max\{\|A\|, \|B\|\}\mu([0,1])$, the binary operation

$$A \,\sigma\, B = \int_{[0,1]} A \,!_t\, B \, d\mu(t), \quad A, B \geq 0$$

is well-defined. The monotonicity (M1) and the transformer inequality (M2) follow from passing these properties of weighted harmonic means through the integral. To show (M3), let $A_n \downarrow A$ and $B_n \downarrow B$. Then $A_n \,!_t\, B_n \downarrow A \,!_t\, B$ for $t \in [0,1]$ by the joint-continuity of weighted harmonic means. We obtain from the dominated convergence theorem that for each $\xi \in H$

$$\begin{aligned}\lim_{n\to\infty} \langle (A_n \,\sigma\, B_n)\xi, \xi\rangle &= \lim_{n\to\infty} \langle \int A_n \,!_t\, B_n \, d\mu(t)\xi, \xi\rangle \\ &= \lim_{n\to\infty} \int \langle (A_n \,!_t\, B_n)\xi, \xi\rangle \, d\mu(t)\end{aligned}$$

$$= \int \langle (A \,!_t\, B)\xi, \xi \rangle \, d\mu(t)$$
$$= \langle \int A \,!_t\, B \, d\mu(t)\xi, \xi \rangle,$$

i.e. $A_n \,\sigma\, B_n \downarrow A \,\sigma\, B$. Thus, we have established a well-defined map $\mu \mapsto \sigma$. For injectivity of this map, let μ_1 and μ_2 be finite Borel measures on $[0,1]$ such that

$$\int_{[0,1]} A \,!_t\, B \, d\mu_1(t) = \int_{[0,1]} A \,!_t\, B \, d\mu_2(t), \quad A, B \geq 0.$$

For each $x \geq 0$, define

$$f_i(x) = \int_{[0,1]} 1 \,!_t\, x \, d\mu_i(t) = \int_{[0,\infty]} \frac{x(\lambda+1)}{x+\lambda} \, d\mu_i \Psi(\lambda) \quad i = 1, 2$$

where $\Psi : [0,\infty] \to [0,1]$, $t \mapsto t/(t+1)$ and the measure $\mu_i \Psi$ is defined for each Borel set E by $E \mapsto \mu_i(\Psi(E))$. Then for each $x \geq 0$, we have $f_1(x) = f_2(x)$ by setting $A = I$ and $B = xI$. Theorem 2.1 implies that $\mu_1 = \mu_2$.

For surjectivity of this map, consider a connection σ. By Theorem 1.1, there is a unique operator monotone function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f(x)I = I \,\sigma\, (xI)$ for $x \geq 0$. By Theorem 2.1, there is a finite Borel measure ν on $[0,\infty]$ such that (2.1) holds. Define a finite Borel measure μ on $[0,1]$ by $\mu = \nu \Psi^{-1}$. Consider a connection η defined by

$$A \,\eta\, B = \int_{[0,1]} A \,!_t\, B \, d\mu(t), \quad A, B \geq 0.$$

Then $I \,\eta\, (xI) = f(x)I$ for $x \geq 0$. Theorem 1.1 implies that $\eta = \sigma$. Hence the map $\mu \mapsto \sigma$ is surjective. This map is affine since μ is finite. It is easy to see that this map is order-preserving. The proof of Theorem 1.3 is therefore complete.

From Theorem 1.3, every connection σ takes the form

$$A \,\sigma\, B = \int_{[0,1]} A \,!_t\, B \, d\mu(t), \quad A, B \geq 0$$

for a unique finite Borel measure μ on $[0,1]$. Note that the 0-weighted harmonic mean and the 1-weighted harmonic mean are the left-trivial mean and the right-trivial mean, respectively. We call the measure μ corresponding to a connection σ the *associated measure* of σ. The relationship between the associated measure μ and the representing measure ν of a connection is given by $\mu = \nu \Psi^{-1}$.

EXAMPLE 2.2. The associated measure of the t-weighted harmonic mean $!_t$ is the Dirac measure δ_t at t. In particular, the associated measures of the left-trivial mean and the right-trivial mean are δ_0 and δ_1, respectively. By affinity of the map

$\mu \mapsto \sigma$, the associated measures of the sum and the parallel sum are given by $\delta_0 + \delta_1$ and $\frac{1}{2}\delta_{1/2}$, respectively. Similarly, the α-weighted arithmetic mean has the associated measure $(1-\alpha)\delta_0 + \alpha\delta_1$. More generally, the measure $\sum_{i=1}^n a_i \, \delta_{t_i}$, where $t_i \in [0,1]$ and $a_i \geq 0$, is associated to the connection $\sum_{i=1}^n a_i \, !_{t_i}$. In particular, the probability measure $(1-\alpha)\delta_t + \alpha\delta_s$, when $\alpha, t, s \in [0,1]$, is associated to the α-weighted arithmetic mean between the t-weighted harmonic mean and the s-weighted harmonic mean.

Example 2.3. Consider the associated measure of the α-weighted geometric mean $\#_\alpha$ for $0 < \alpha < 1$. From contour integrals, its representing function is given by

$$x^\alpha = \int_{[0,\infty]} \frac{x\lambda^{\alpha-1}}{x+\lambda} \cdot \frac{\sin \alpha\pi}{\pi} \, dm(\lambda).$$

It follows that the associated measure of $\#_\alpha$ is $d\mu(t) = \dfrac{\sin \alpha\pi}{\pi} \cdot \dfrac{1}{t^{1-\alpha}(1-t)^\alpha} \, dt.$

Example 2.4. Let us compute the associated measure of the logarithmic mean. Recall that the representing function of this mean is the operator monotone function $f(x) = (x-1)/\log x$. Then, by Example 2.3, we have

$$\begin{aligned} f(x) = \int_0^1 x^\lambda \, d\lambda \; &= \int_0^1 \int_0^1 \frac{\sin \lambda\pi}{\pi} \cdot \frac{1 \, !_t \, x}{t^{1-\lambda}(1-t)^\lambda} \, dt \, d\lambda \\ &= \int_0^1 (1 \, !_t \, x) \int_0^1 \frac{\sin \lambda\pi}{\pi t^{1-\lambda}(1-t)^\lambda} \, d\lambda \, dt. \end{aligned}$$

Hence the associated measure has density g given by

$$g(t) = \int_0^1 \frac{\sin \lambda\pi}{\pi t^{1-\lambda}(1-t)^\lambda} \, d\lambda = \frac{1}{t(1-t)\left(\pi^2 + \log^2(\frac{t}{1-t})\right)}.$$

Example 2.5. Consider the dual of the logarithmic mean defined by

$$A \, \eta \, B = \{LM(B^{-1}, A^{-1})\}^{-1},$$

where LM denotes the logarithmic mean. It turns out that its associated measure is Lebesgue measure on $[0,1]$. Indeed, its representing function is given by $x \mapsto \frac{x}{x-1} \log x$. We thus have the integral representation

$$A \, \eta \, B = \int_{[0,1]} A \, !_t \, B \, dt.$$

Remark 2.6. Even though the map $\mu \mapsto \sigma$ is order-preserving, the inverse map $\sigma \mapsto \mu$ is not order-preserving in general. For example, the associated measures of

the harmonic mean $!_{1/2}$ and the arithmetic mean $\nabla_{1/2} = (!_0 + !_1)/2$ are given by $\delta_{1/2}$ and $(\delta_0 + \delta_1)/2$, respectively. We have $!_{1/2} \leq \nabla_{1/2}$ but $\delta_{1/2} \not\leq (\delta_0 + \delta_1)/2$.

Corollary 2.7. *A connection is a mean if and only if its associated measure is a probability measure.*

Proof. Let σ be a connection with representing function f and associated measure μ. Recall that σ is a mean if and only if $f(1) = 1$. From the integral representation (1.2), we have $f(1)I = I\,\sigma\,I = \mu([0,1])I$. □

Corollary 2.8. *There is a one-to-one correspondence between means and probability Borel measures on the unit interval. In fact, every mean takes the form*

$$A\,\sigma\,B = \int_{[0,1]} A\,!_t\,B\,d\mu(t), \quad A, B \geq 0$$

for a unique probability Borel measure μ on $[0,1]$.

Hence every mean can be regarded as an average of special means, namely, weighted harmonic means. The weighted harmonic means form building blocks for general means.

Corollary 2.9. *The weighted harmonic means are the extreme points of the convex set of means.*

Proof. Use Theorem 1.3, Corollary 2.7 and the fact that the Dirac measures are the extreme points of the convex set of probability Borel measures on $[0,1]$. □

The transpose of a connection σ is the connection $(A,B) \mapsto B\,\sigma\,A$. Hence, a connection is symmetric if and only if it coincides with its transpose. It was shown in [9] that if f is the representing function of σ, then the representing function of the transpose of σ is given by $x \mapsto xf(1/x)$ for $x > 0$.

Theorem 2.10. *Let σ be a connection with associated measure μ. Then*

- *the representing function of the transpose of σ is given by*

$$x \mapsto \int_{[0,1]} x\,!_t\,1\,d\mu(t), \quad x \geq 0; \tag{2.2}$$

- *the associated measure of the transpose of σ is given by $\mu\Theta^{-1} = \mu\Theta$ where $\Theta : [0,1] \to [0,1], t \mapsto 1-t$.*

Proof. The set function $\mu\Theta$ is clearly a finite Borel measure on $[0,1]$. Since the representing function f of σ is given by (1.3), the representing function of the transpose of σ is given by

$$xf(\frac{1}{x}) = x\int_{[0,1]} 1\,!_t\,\frac{1}{x}\,d\mu(t) = \int_{[0,1]} x\,!_t\,1\,d\mu(t)$$

for each $x > 0$. By continuity, the representing function of the transpose of σ is given by (2.2). By changing the variable $t \mapsto 1-t$ and Theorem 1.3, we obtain that the associated measure of the transpose of σ is $\mu\Theta$. □

We say that a Borel measure μ on $[0,1]$ is *symmetric* if $\mu\Theta = \mu$.

COROLLARY 2.11. *There is a one-to-one correspondence between symmetric connections and finite symmetric Borel measures on* $[0,1]$ *via the integral representation*

$$A \,\sigma\, B = \frac{1}{2}\int_{[0,1]} (A \,!_t\, B) + (B \,!_t\, A)\, d\mu(t), \quad A, B \geq 0$$

The representing function of the connection σ in (2.11) can be written by

$$f(x) = \frac{1}{2}\int_{[0,1]} (1 \,!_t\, x) + (x \,!_t\, 1)\, d\mu(t), \quad x \geq 0 \tag{2.3}$$

and its associated measure is μ. In particular, a connection is symmetric if and only if its associated measure is symmetric.

Proof. It follows from Theorems 1.3 and 2.10. □

COROLLARY 2.12. *There is a one-to-one correspondence between symmetric means and probability symmetric Borel measures on the unit interval. In fact, every symmetric mean takes the form*

$$A \,\sigma\, B = \frac{1}{2}\int_{[0,1]} (A \,!_t\, B) + (B \,!_t\, A)\, d\mu(t), \quad A, B \geq 0$$

for a unique probability symmetric Borel measure μ on $[0,1]$.

Proof. Use Corollaries 2.7 and 2.11. □

3. Decompositions of connections. This section deals with decompositions of connections, means, symmetric connections and symmetric means.

THEOREM 3.1. *Let σ be a connection on $B(H)^+$. Then there is a unique triple $(\sigma_{ac}, \sigma_{sc}, \sigma_{sd})$ of connections on $B(H)^+$ such that*

$$\sigma = \sigma_{ac} + \sigma_{sc} + \sigma_{sd}$$

and

(i) σ_{sd} is a countable sum of weighted harmonic means with nonnegative coefficients, i.e. there are a countable set $D \subseteq [0,1]$ and a family $\{a_t\}_{t\in D} \subseteq \mathbb{R}^+$ such that $\sum_{t\in D} a_t < \infty$ and

$$\sigma_{sd} = \sum_{t\in D} a_t \,!_t,$$

i.e. for each $A, B \geq 0$, $A\,\sigma_{sd}\,B = \sum_{t \in D} a_t (A\,!_t B)$ and the series converges in the norm topology;

(ii) there is a (unique m-a.e.) integrable function $g : [0,1] \to \mathbb{R}^+$ such that

$$A\,\sigma_{ac}\,B = \int_{[0,1]} g(t)(A\,!_t\,B)\,dm(t), \quad A, B \geq 0;$$

(iii) its associated measure of σ_{sc} is continuous and mutually singular to m.

Moreover, the representing functions of σ_{ac}, σ_{sd} and σ_{sc} are given respectively by

$$f_{sd}(x) = \int_{[0,1]} 1\,!_t\,x\,d\mu_{sd} = \sum_{t \in D} a_t (1\,!_t\,x) \tag{3.1}$$

$$f_{ac}(x) = \int_{[0,1]} 1\,!_t\,x\,d\mu_{ac}, \tag{3.2}$$

$$f_{sc}(x) = \int_{[0,1]} 1\,!_t\,x\,d\mu_{sc} \tag{3.3}$$

and the associated measure of σ_{sd} is given by $\sum_{t \in D} a_t\,\delta_t$.

Proof. Let μ be the associated measure of σ. Then there is a unique triple $(\mu_{ac}, \mu_{sc}, \mu_{sd})$ of finite Borel measures on $[0,1]$ such that $\mu = \mu_{ac} + \mu_{sc} + \mu_{sd}$ where μ_{sd} is a discrete measure, μ_{ac} is absolutely continuous with respect to m and μ_{sc} is a continuous measure mutually singular to m. Define $\sigma_{ac}, \sigma_{sc}, \sigma_{sd}$ to be the connections associated to $\mu_{ac}, \mu_{sc}, \mu_{sd}$, respectively. Then

$$A\,\sigma_{sd}\,B = \int_{[0,1])} A\,!_t\,B\,d\mu_{sd}(t) = \sum_{t \in D} a_t (A\,!_t\,B), \quad A, B \geq 0,$$

where the series $\sum_{t \in D} a_t (A\,!_t\,B)$ converges in norm. Indeed, the fact that, for each $n < m$ in N and $t_i \in [0,1]$,

$$\begin{aligned}
\left\| \sum_{i=1}^{n} a_{t_i}(A\,!_{t_i}\,B) - \sum_{i=1}^{m} a_{t_i}(A\,!_{t_i}\,B) \right\| &\leq \sum_{i=n+1}^{m} a_{t_i} \|A\,!_{t_i}\,B\| \\
&\leq \sum_{i=n+1}^{m} a_{t_i} (\|A\|\,!_{t_i}\,\|B\|) \\
&\leq \sum_{i=n+1}^{m} a_{t_i} \max\{\|A\|, \|B\|\}
\end{aligned}$$

together with the condition $\sum_{i=1}^{\infty} a_{t_i} < \infty$ implies the convergence of the series $\sum_{i=1}^{\infty} a_{t_i}(A\,!_{t_i}\,B)$. Using Theorem 1.1, the representing functions of $\sigma_{sd}, \sigma_{ac}, \sigma_{sc}$ are given by f_{sd}, f_{ac}, f_{sc} in (3.1), (3.2), (3.3), respectively. The condition (i) comes from the fact that the associated measure of $!_t$ is δ_t for each $t \in [0,1]$ in Example 2.2.

The condition (ii) follows from Radon-Nikodym theorem. The uniqueness of such decomposition is obtained from the uniqueness of the decomposition of measures. ∎

This theorem says that every connection σ consists of three parts. The "singular discrete part" σ_{sd} is a countable sum of weighted harmonic means. Such type of connections include the weighted arithmetic means, the sum and the parallel sum. The "absolutely continuous part" σ_{ac} arises as an integral with respect to Lebesgue measure, given by the formula (3.1). Examples 2.3, 2.4 and 2.5 show that the weighted geometric means, the logarithmic mean and its dual are typical examples of such connections. The "singular continuous part" σ_{sc} has an integral representation with respect to a continuous measure mutually singular to Lebesgue measure. Hence (aside singular continuous part) this theorem gives an explicit description of general connections.

Proposition 3.2. *Consider the connection σ_{ac} defined by (3.1). Then*

- *it is a mean if and only if the average of the density function g is 1, i.e. $\int_0^1 g(t)\,dt = 1$.*
- *it is a symmetric connection if and only if $g \circ \Theta = g$.*

Proof. Use Corollaries 2.7 and 2.11. ∎

We say that a density function $g : [0,1] \to \mathbb{R}^+$ is *symmetric* if $g \circ \Theta = g$. Next, we decompose a mean as a convex combination of means.

Corollary 3.3. *Let σ be a mean on $B(H)^+$. Then there are unique triples $(\widetilde{\sigma_{ac}}, \widetilde{\sigma_{sc}}, \widetilde{\sigma_{sd}})$ of means or zero connections on $B(H)^+$ and (k_{ac}, k_{sc}, k_{sd}) of real numbers in $[0,1]$ such that*

$$\sigma = k_{ac}\widetilde{\sigma_{ac}} + k_{sc}\widetilde{\sigma_{sc}} + k_{sd}\widetilde{\sigma_{sd}}, \quad k_{ac} + k_{sc} + k_{sd} = 1$$

and

(i) there are a countable set $D \subseteq [0,1]$ and a family $\{a_t\}_{t\in D} \subseteq \mathbb{R}^+$ such that $\sum_{t\in D} a_t = 1$ and $\widetilde{\sigma_{sd}} = \sum_{t\in D} a_t \,!_t$;

(ii) there is a (unique m-a.e.) integrable function $g : [0,1] \to \mathbb{R}^+$ with average 1 such that

$$A \,\widetilde{\sigma_{ac}}\, B = \int_{[0,1]} g(t)(A \,!_t\, B)\,dm(t), \quad A, B \geq 0;$$

(iii) its associated measure of $\widetilde{\sigma_{sc}}$ is continuous and mutually singular to m.

Proof. Let μ be the associated probability measure of $\sigma = \sigma_{ac} + \sigma_{sd} + \sigma_{sc}$ and write $\mu = \mu_{ac} + \mu_{sd} + \mu_{sc}$. Suppose that μ_{ac}, μ_{sd} and μ_{sc} are nonzero. Then

$$\mu = \mu_{ac}([0,1])\frac{\mu_{ac}}{\mu_{ac}([0,1])} + \mu_{sd}([0,1])\frac{\mu_{sd}}{\mu_{sd}([0,1])} + \mu_{sc}([0,1])\frac{\mu_{sc}}{\mu_{sc}([0,1])}.$$

Set

$$\widetilde{\mu_a} = \frac{\mu_{ac}}{\mu_{ac}([0,1])}, \quad \widetilde{\mu_{sd}} = \frac{\mu_{sd}}{\mu_{sd}([0,1])}, \quad \widetilde{\mu_{sc}} = \frac{\mu_{sc}}{\mu_{sc}([0,1])},$$
$$k_{ac} = \mu_{ac}([0,1]), \quad k_{sd} = \mu_{sd}([0,1]), \quad k_{sc} = \mu_{sc}([0,1]).$$

Define $\widetilde{\sigma_{ac}}, \widetilde{\sigma_{sd}}, \widetilde{\sigma_{sc}}$ to be the means corresponding to the measures $\widetilde{\mu_{ac}}, \widetilde{\mu_{sd}}, \widetilde{\mu_{sc}}$, respectively. Now, apply Theorem 3.1 and Proposition 3.2. □

We can decompose a symmetric connection as a nonnegative linear combination of symmetric connections as follows:

COROLLARY 3.4. *Let σ be a symmetric connection on $B(H)^+$. Then there is a unique triple $(\sigma_{ac}, \sigma_{sc}, \sigma_{sd})$ of symmetric connections on $B(H)^+$ such that*

$$\sigma = \sigma_{ac} + \sigma_{sc} + \sigma_{sd}$$

and

(i) there are a countable set $D \subseteq [0,1]$ and a family $\{a_t\}_{t\in D} \subseteq \mathbb{R}^+$ such that $a_t = a_{1-t}$ for all $t \in D$, $\sum_{t\in D} a_t < \infty$ and $\sigma_{sd} = \sum_{t\in D} a_t \,!_t$;

(ii) there is a (unique m-a.e.) symmetric integrable function $g : [0,1] \to \mathbb{R}^+$ such that

$$A \,\sigma_{ac}\, B = \frac{1}{2}\int_{[0,1]} g(t)(A \,!_t\, B + B \,!_t\, A)\, dm(t), \quad A, B \geq 0;$$

(iii) its associated measure of σ_{sc} is continuous and mutually singular to m.

Proof. Let μ be the associated measure of σ and decompose $\mu = \mu_{ac} + \mu_{sd} + \mu_{sc}$ where $\mu_{ac} \ll m$, μ_{sd} is a discrete measure, μ_{sc} is continuous and $\mu_{sc} \perp m$. Then $\mu\Theta = \mu_{ac}\Theta + \mu_{sd}\Theta + \mu_{sc}\Theta$. It is straightforward to show that $\mu_{ac}\Theta \ll m$, $\mu_{sd}\Theta$ is discrete, $\mu_{sc}\Theta$ is continuous and $\mu_{sc}\Theta \perp m$. By Corollary 2.11, $\mu\Theta = \mu$. The uniqueness of the decomposition of measures implies that $\mu_{ac}\Theta = \mu_{ac}, \mu_{sd}\Theta = \mu_{sd}$ and $\mu_{sc}\Theta = \mu_{sc}$. Again, Corollary 2.11 tells us that σ_{ac}, σ_{sd} and σ_{sc} are symmetric connections. The rest of the proof follows from Theorem 3.1 and Proposition 3.2. □

A decomposition of a symmetric mean as a convex combination of symmetric means is obtained by normalizing symmetric connections as in the proof of Corollary 3.3.

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